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The van Kampen Expansion for the Fokker–Planck Equation of a Duffing Oscillator

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In Rodríguez and van Kampen's 1976 paper a method of extracting information from the Fokker–Planck equation without having to solve the equation is outlined. The Fokker–Planck equation for a Duffing oscillator excited by white noise is expanded about the intensity α of the forcing function. This expansion is carried to order $\mathcal{O}(\alpha^{1/2})$. However, no studies are made of the effects of the order of the expansion or variation of the parameters, nor are comparisons made to experimental results. In the present paper, the expansion is carried to a higher order, $\mathcal{O}(\alpha^{3/2})$, results are presented and compared to Monte Carlo experiments using both white and colored noise, and parametric studies are performed on the intensity of the forcing function and the damping coefficient. It is found that the expansion method works well for the case of white noise and for colored noise where the correlation time is less than 0.1 sec, but fails to give certain details. It is also found that the system behaves as expected when the parameters are varied.

KEY WORDS: Fokker–Planck equation; white noise; colored noise; van Kampen expansion; Monte Carlo; Duffing oscillator.

1. INTRODUCTION

The Fokker–Planck equation has proven to be a useful tool in the analysis of simple nonlinear oscillators excited by stochastic processes. As a partial differential equation for the probability density function of the response, its solution completely defines the solution of the problem. It can be used to analyze both a single oscillator of the form

$$m\ddot{x} + \gamma(\dot{x}, x)\dot{x} + k(\dot{x}, x)x = \mathcal{F}(t) \tag{1}$$

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or a system of multiple, linked, oscillators of the form

$$\mathbf{M}\ddot{\underline{x}} + \mathbf{\Gamma}(\dot{\underline{x}}, \underline{x})\dot{\underline{x}} + \mathbf{K}(\dot{\underline{x}}, \underline{x})\underline{x} = \underline{\mathcal{F}}(t) \tag{2}$$

In many cases, a physical system can be approximated by such a system of nonlinear oscillators. The systems so modeled can range from a Brownian particle to structures excited by von Kármán vortex shedding. Such modeling can be useful for gaining insight into a problem and the way in which the system will behave as certain parameters are varied.

Once one has decided on the system of oscillators to be used to represent the physical system, the derivation of the Fokker–Planck equation is relatively straightforward, although tedious. The problem remains of how to solve it for the probability distribution of the response. In a very few cases, the Fokker–Planck equation can be solved analytically, but in most cases no analytical solution exists and one usually must resort to a numerical solution. However, this can be computationally intensive and gives little insight into the larger problem.

In their 1976 paper, Rodríguez and van Kampen[1] outline a method of dealing with the case of an oscillator excited by weak Gaussian white noise. The Fokker–Planck equation of the system is expanded about the intensity α of the driving function. This expansion is carried to $\mathcal{O}(\alpha^{1/2})$. In this way the statistics of the fluctuations are obtained directly. This method shows promise as a way to use the Fokker–Planck equation to gain useful information about a wider variety of systems than was possible before.

This is the first of a planned series of papers exploring the usefulness of this method. As in the original paper, the method is applied to the problem of a Duffing oscillator excited by Gaussian white noise. The inherent assumptions of the method are explained here in detail. The expansion is carried both to the same order as in the original paper and to $\mathcal{O}(\alpha^{3/2})$. Results are presented and compared to a Monte Carlo experiment. Parametric studies are done on the parameter of expansion as well as on the other important variable in the expansion: the coefficient of damping.

2. EXPANSION OF THE FOKKER–PLANCK EQUATION FOR A DUFFING OSCILLATOR

As in the Rodríguez and van Kampen paper, henceforth called the "original paper," the system under consideration is a Duffing oscillator in a heat bath. The equation of motion can be written simply as

$$\ddot{x} + \gamma\dot{x} + x + x^3 = F(t) \tag{3}$$

$F(t)$ is a Langevin force[2] and is assumed to be Gaussian white noise with the following properties:

$$\begin{aligned} \langle F(t) \rangle &= 0 \\ \langle F(t)\, F(t') \rangle &= 2\alpha\delta(t-t') \end{aligned} \tag{4}$$

It is assumed that the immersion of the oscillation takes place at time $t=0$ and that the system is not necessarily at rest.

$f(x, v; t)$ is defined as the joint probability density function of x and $v=\dot{x}$ at time t. This leads to the following Fokker–Planck equation (see Ochi[3] for a complete derivation):

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - (x+x^3)\frac{\partial f}{\partial v} = \gamma\frac{\partial}{\partial v}(vf) + \alpha\frac{\partial^2 f}{\partial v^2} \tag{5}$$

As $f(x, v; t)$ is a complete description of the system response, solution of the above differential equation for f constitutes a solution of the problem.

The same assumptions on the sizes of the variables in Eq. (5) are made as in the original paper: namely, that γ, x, and v will all be assumed to be of the same order of magnitude, and much larger than α.

In the original paper, it is assumed that the system response due to the forcing function will be small compared to the deterministic response due to the initial conditions. Therefore, the total response can be viewed as random fluctuations Δ_x and Δ_v superimposed onto the deterministic response to the initial conditions. Furthermore, the random fluctuations will be of order $\alpha^{1/2}$. Because the only source of energy is $F(t)$, the power input to the system is proportional to α. But if the system is to remain stable, then the viscous power dissipation of the fluctuations caused by the influence of $F(t)$ must be of equal average magnitude as the power input. Therefore $\mathcal{O}(\gamma\eta^2) = \mathcal{O}(\alpha)$ or $\mathcal{O}(\eta) = \mathcal{O}(\alpha^{1/2})$. But the kinetic energy of the fluctuations must be of the same order as the potential energy. So $\mathcal{O}(\zeta^2) = \mathcal{O}(\eta^2)$, which implies that $\mathcal{O}(\zeta) = \mathcal{O}(\alpha^{1/2})$. Therefore, $\Delta_x = \alpha^{1/2}\zeta$ and $\Delta_v = \alpha^{1/2}\eta$, where ζ and η are of order unity. Therefore the following substitutions are made:

$$x = \phi(t) + \alpha^{1/2}\zeta \tag{6}$$

$$v = \psi(t) + \alpha^{1/2}\eta \tag{7}$$

In the original paper, the initial conditions $\phi(0)$ and $\psi(0)$ are assumed to be zero and the expansion carried through to give the time derivatives of the second-order moments. In Weinstein and Benaroya[4] the expansion is explained in somewhat greater detail and the fourth-order moments are also derived. In total the following eight equations are derived:

$$\frac{d}{dt}\langle\zeta^2\rangle = 2\langle\zeta\eta\rangle \tag{8}$$

$$\frac{d}{dt}\langle\zeta\eta\rangle = \langle\eta^2\rangle - \langle\zeta^2\rangle - \gamma\langle\zeta\eta\rangle - \alpha\langle\zeta^4\rangle + \mathcal{O}(\alpha^{3/2}) \tag{9}$$

$$\frac{d}{dt}\langle\eta^2\rangle = -2\langle\zeta\eta\rangle - 2\gamma\langle\eta^2\rangle - 2\alpha\langle\zeta^3\eta\rangle + 2 + \mathcal{O}(\alpha^{3/2}) \tag{10}$$

$$\frac{d}{dt}\langle\zeta^4\rangle = 4\langle\zeta^3\eta\rangle \tag{11}$$

$$\frac{d}{dt}\langle\zeta^3\eta\rangle = 3\langle\zeta^2\eta^2\rangle - \langle\zeta^3\eta\rangle - \langle\zeta^4\rangle + \mathcal{O}(\alpha^{3/2}) \tag{12}$$

$$\frac{d}{dt}\langle\zeta^2\eta^2\rangle = 2\langle\zeta\eta^3\rangle - 2\gamma\langle\zeta^2\eta^2\rangle - 2\langle\zeta^3\eta\rangle + 2\langle\zeta^2\rangle + \mathcal{O}(\alpha^{3/2}) \tag{13}$$

$$\frac{d}{dt}\langle\zeta\eta^3\rangle = \langle\eta^4\rangle - 3\gamma\langle\zeta\eta^3\rangle - 3\langle\zeta^2\eta^2\rangle + 6\langle\zeta\eta\rangle + \mathcal{O}(\alpha^{3/2}) \tag{14}$$

$$\frac{d}{dt}\langle\eta^4\rangle = -4\gamma\langle\eta^4\rangle - 4\langle\zeta\eta^3\rangle + 12\langle\eta^2\rangle + \mathcal{O}(\alpha^{3/2}) \tag{15}$$

If it is the stationary behavior that is of interest, then one can obtain the equilibrium values of these quantities by setting each time derivative equal to zero. Then the equilibrium, or stationary, states of each expectation can be found by simple linear algebra:

$$\langle\zeta^2\rangle_{eq} = \frac{1}{\gamma} - \alpha\frac{3}{\gamma^2} \qquad \langle\zeta^3\eta\rangle_{eq} = 0 + \mathcal{O}(\alpha^{3/2})$$

$$\langle\zeta\eta\rangle_{eq} = 0 \qquad \langle\zeta^2\eta^2\rangle_{eq} = \frac{1}{\gamma^2} + \mathcal{O}(\alpha^{3/2})$$

$$\langle\eta^2\rangle_{eq} = \frac{1}{\gamma} + \mathcal{O}(\alpha^{1/2}) \qquad \langle\zeta\eta^3\rangle_{eq} = 0 + \mathcal{O}(\alpha^{3/2})$$

$$\langle\zeta^4\rangle_{eq} = \frac{3}{\gamma^2} + \mathcal{O}(\alpha^{3/2}) \qquad \langle\eta^4\rangle_{eq} = \frac{3}{\gamma^2} + \mathcal{O}(\alpha^{3/2})$$

If one is interested in the transient response of the system, and if one can accept a solution of $\mathcal{O}(\alpha^{1/2})$, then one can obtain it analytically. Equations (8)–(10) can be written in matrix form as follows:

$$\frac{d}{dt}\begin{pmatrix}\langle\langle\zeta^2\rangle\rangle\\ \langle\langle\zeta\eta\rangle\rangle\\ \langle\langle\eta^2\rangle\rangle\end{pmatrix} = \mathcal{M}_3\begin{pmatrix}\langle\langle\zeta^2\rangle\rangle\\ \langle\langle\zeta\eta\rangle\rangle\\ \langle\langle\eta^2\rangle\rangle\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix} + \mathcal{O}(\alpha^{3/2}) \tag{16}$$

where

$$\mathcal{M}_3 = \begin{Bmatrix} 0 & 2 & 0 \\ -1 & -\gamma & 1 \\ 0 & -2 & -2\gamma \end{Bmatrix}$$

This can be solved through standard techniques to yield the time-evolving variances. It must be noted that by solving only these three equations, the order of the solution has been reduced to $\mathcal{O}(\alpha^{1/2})$:

$$\begin{pmatrix} \ll\zeta^2\gg \\ \ll\zeta\eta\gg \\ \ll\eta^2\gg \end{pmatrix} = \begin{Bmatrix} 1/\lambda_1 & -1/2\lambda_2 & -1/\lambda_3 \\ 1/2 & -1/4 & -1/4 \\ 1/\lambda_1 & -1/\lambda_2 & -1/2\lambda_3 \end{Bmatrix} \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ e^{\lambda_3 t} \end{pmatrix} + \begin{pmatrix} 1/\gamma \\ 0 \\ 1/\gamma \end{pmatrix} \tag{17}$$

where

$$\lambda_1 = -\gamma$$
$$\lambda_2 = -\gamma + 2iw$$
$$\lambda_3 = -\gamma - 2iw$$
$$w = (1 - \tfrac{1}{4}\gamma^2)^{1/2}$$

If one is interested in the transient response and desires a solution of order $\mathcal{O}(\alpha^{3/2})$, one could cast all of Eqs. (8)–(15) into a matrix equation of the form of Eq. (16). This would yield a matrix of rank 8. To solve this analytically would require the analytical eigensolution of this rank-8 matrix. This is a difficult proposition at best with unclear practical need. Instead, for each particular set of parameters γ and μ a numerical matrix is obtained. The eigensolution is then obtained numerically and the results calculated. It should be noted that there is no theoretical loss of accuracy in solving the system of equations this way; only roundoff error degrades the accuracy of the solution.

Thus we have a method for obtaining in closed form the time-evolving moments of the Duffing oscillator subjected to a white-noise forcing function.

3. RESULTS

The response of a Duffing oscillator with damping coefficient $\gamma = 1.0$ excited by white noise of intensity $\alpha = 0.1$ was calculated using Eq. (17) and by solution of all of Eqs. (8)–(15). As a point of comparison, a Monte Carlo experiment simulating a Duffing oscillator with the same parameters

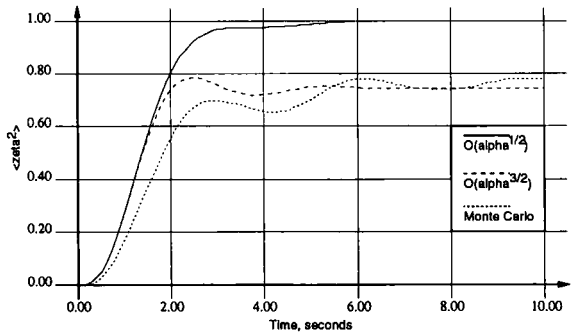

Fig. 1. Time evolution of $\langle \zeta^2 \rangle$.

was performed. This Monte Carlo experiment consisted of 1000 iterations of a fourth-order Runge–Kutta integration of the following restatement of Eqs. (3):

$$\frac{d}{dt}x = v, \qquad \frac{d}{dt}v = \mathcal{F}(t) - v - x - x^3 \tag{18}$$

The results of the $\mathcal{O}(\alpha^{1/2})$ analysis, the $\mathcal{O}(\alpha^{3/2})$ analysis, and the Monte Carlo experiment are plotted in Figs. 1–3.

Figure 1 shows the time evolution of $\langle \zeta^2 \rangle$ as calculated by all three methods. The $\mathcal{O}(\alpha^{1/2})$ analysis shows $\langle \zeta^2 \rangle$ increasing monotonically to its steady-state value of approximately one. The higher-order analysis shows $\langle \zeta^2 \rangle$ increasing in nearly monotonic fashion to its steady-state value of about 0.75. However, this curve does exhibit some overshoot at about 2.5 sec. The results of the Monte Carlo analysis are quite close to those of

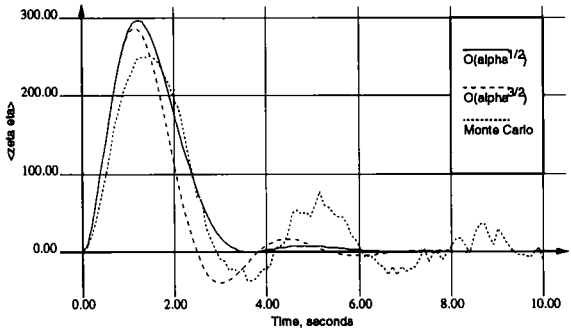

Fig. 2. Time evolution of $\langle \zeta\eta \rangle$.

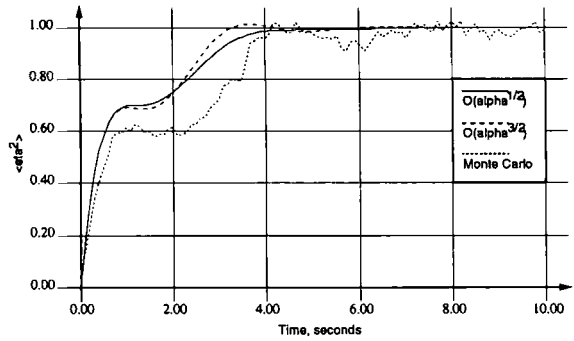

Fig. 3. Time evolution of $\langle \eta^2 \rangle$.

the higher-order analysis, reaching the same steady-state value. The main difference between the two curves is the slightly greater rise time of the Monte Carlo results. The higher limit of the $\mathcal{O}(\alpha^{1/2})$ analysis and the overshoot of the $\mathcal{O}(\alpha^{3/2})$ analysis are artifacts of the expansion process. The higher-order, $\mathcal{O}(\alpha^{3/2})$, time derivatives differ from the lower-order, $\mathcal{O}(\alpha^{1/2})$, ones by the subtraction of $\alpha\langle\zeta^4\rangle$ in the case of $(d/dt)\langle\zeta\eta\rangle$ and of $2\alpha\langle\zeta^3\eta\rangle$ in the case of $(d/dt)\langle\zeta^2\rangle$. However, it can be seen from Fig. 4 that $\langle\zeta^4\rangle \geqslant 0$ during the entire time span of interest and rises in magnitude quickly, and $\langle\zeta^3\eta\rangle$ is greater than zero during the first half of the time span of interest, where the second-order moments are changing most rapidly. This would explain the slower rise in the magnitude of the $\mathcal{O}(\alpha^{3/2})$ solutions than the $\mathcal{O}(\alpha^{1/2})$ solutions. It also indicates that inclusion of the higher-order terms serves to lower the overall value of the analytical results and that the omission thereof causes the higher overall values of the analytical methods than those of the Monte Carlo methods. As one would expect, the higher-order

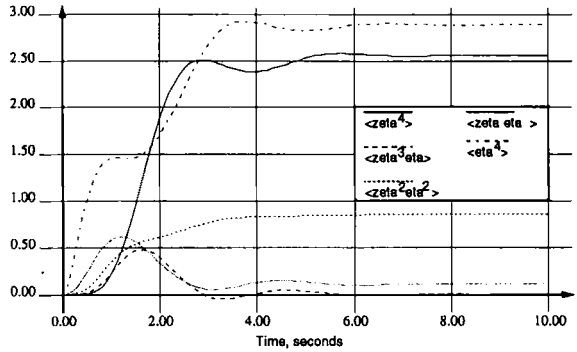

Fig. 4. The time evolution of the fourth-order moments for $\gamma = 1.0$, $\alpha = 0.1$.

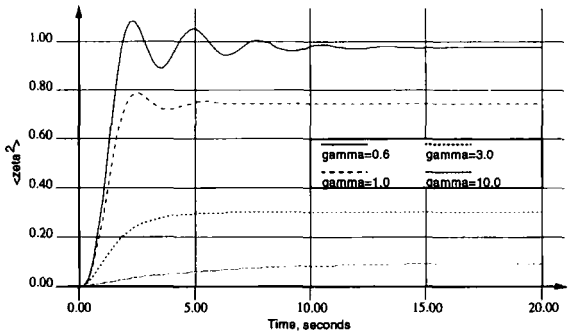

Fig. 5. $\langle \zeta^2 \rangle$ versus time for increasing values of γ, $\alpha = 0.1$, calculated analytically to $\mathcal{O}(\alpha^{2/2})$.

analyses consistently show more points of inflection than do the lower-order analyses.

Figure 2 shows nearly identical curves for all methods. As in Fig. 1, the $\mathcal{O}(\alpha^{3/2})$ analysis shows more overshoot in one place than does the lower-order analysis. Here this overshoot occurs at about 4 sec, at the second local extreme. It is also noted that only the $\mathcal{O}(\alpha^{3/2})$ method accurately reflects the region where the Monte Carlo curve is negative. The initial excursion of the Monte Carlo curve is not as great as that of the other two curves, although all three approach the x axis as time increases.

Figure 3 provides a qualitatively similar comparison of results with both analyses almost coincidental and again slightly greater in the transient region than the Monte Carlo experiment. All three curves approach a steady-state value of approximately 1.

Figure 5 shows how the time evolution of $\langle \zeta^2 \rangle$, as calculated by the $\mathcal{O}(\alpha^{3/2})$ analysis, is affected by increasing γ. Figure 6 shows the same study for the Monte Carlo experiment. Figure 5 shows not only a decrease in the

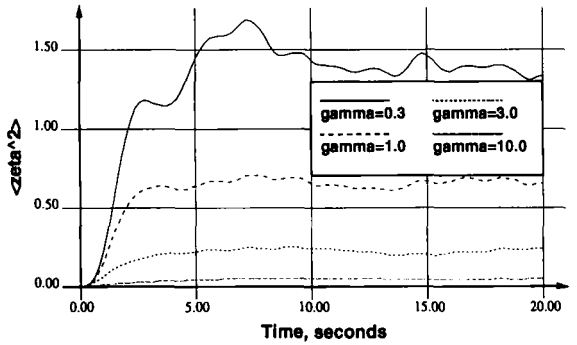

Fig. 6. $\langle \zeta^2 \rangle$ versus time for increasing values of γ, $\alpha = 0.1$, from Monte Carlo analysis.

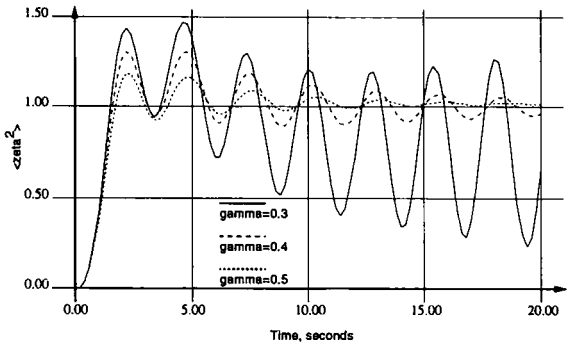

Fig. 7. $\langle \zeta^2 \rangle$ versus time for decreasing values of γ, $\alpha = 0.1$, calculated analytically to $\mathcal{O}(\alpha^{2/2})$.

steady-state values of $\langle \zeta^2 \rangle$ with increasing values of γ, but also a smoothing of the curves. The decrease in the steady-state values is also seen in the Monte Carlo curves. The cause of the decrease is a physical one: increased damping implies increased viscous energy dissipation which leads to smaller excursions of the oscillator. Comparison of Figs. 5 and 6 shows that even when γ becomes large enough to violate the order-unity assumption, the analysis still gives reasonable results. In fact the agreement between results becomes better at higher values of γ. It can be seen that, even for $\gamma = 0.6$, the Monte Carlo curves are relatively smooth, with the exception of the small-scale "wiggles" inherent in Monte Carlo analysis. This indicates that the multiple local maxima and minima, i.e., at 3, 5, 7, and 9 sec, in the analytical curves at $\gamma = 0.6$ are due to the low order of γ of the analysis. It was assumed at the beginning of the analysis that γ is of order unity. However, it can be seen from these curves that the further γ is from this assumption, the worse are the results of the analysis.

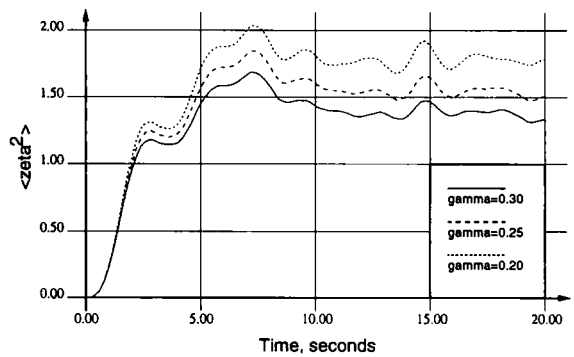

Fig. 8. $\langle \zeta^2 \rangle$ versus time for decreasing values of γ, $\alpha = 0.1$, calculated by Monte Carlo simulation.

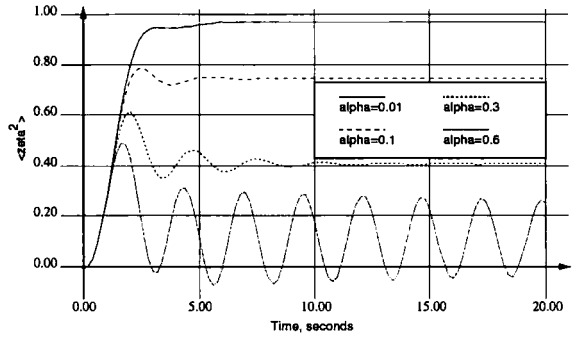

Fig. 9. $\langle \zeta^2 \rangle$ versus time for different values of α, $\gamma = 1.0$, calculated analytically to $\mathcal{O}(\alpha^{2/2})$.

The behavior of the system as γ becomes small is further investigated in Figs. 7 and 8. These curves show how, as γ goes from 0.4 to 0.3, the analysis breaks down completely. It was shown in Figs. 5 and 6 that the $\gamma = 0.6$ analysis differed significantly from the Monte Carlo analysis in the transient region of 0–5 sec, although its large-time behavior was accurate. Figures 7 and 8 show that as γ is made smaller, even the large-time behavior becomes unreasonable, with $\langle \zeta^2 \rangle$ becoming negative, which is clearly impossible.

Figures 9 and 10 show how the time evolutions of $\langle \zeta^2 \rangle$ are affected by increasing values of α. One can see in Fig. 9 what happens as the assumption that α is small as compared to unity is violated. As is shown in Fig. 9, the curve for $\alpha = 0.3$ exhibits oscillations that damp out slowly with time. As α becomes even larger, negative values for $\langle \zeta^2 \rangle$ develop. However, since the average of a squared quantity cannot be negative, these

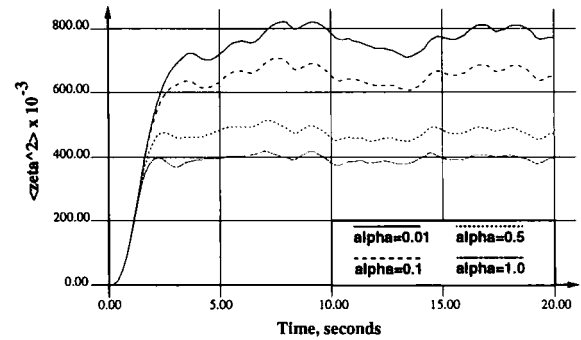

Fig. 10. $\langle \zeta^2 \rangle$ versus time for different values of α, $\gamma = 1.0$, calculated by Monte Carlo simulation.

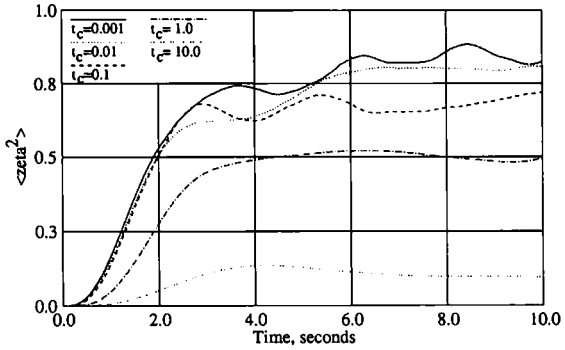

Fig. 11. $\langle \zeta^2 \rangle$ versus time for different values of τ_c, of $\gamma = 1.0$, $\alpha = 0.1$, calculated by Monte Carlo simulation.

results are spurious. The trend of the steady-state value of $\langle \zeta^2 \rangle$ decreasing with increasing values of α is common to both figures. This does not violate the basic assumption that ζ is of order unity when α is small. ζ does indeed remain of order unity; it was not assumed that ζ was independent of α. Here $\langle \zeta^2 \rangle$ decreases with increasing α due to the nonlinearity of the Duffing oscillator. The energy stored in the nonlinear spring is greater than that stored in the linear spring by $\mathscr{E}_{\text{nonlinear}} - \mathscr{E}_{\text{linear}} = \frac{1}{4}x^4$. Therefore the effect of the nonlinearity of the Duffing oscillator is to decrease $\langle \zeta^2 \rangle$, and this effect will increase with increasing $\langle \zeta^2 \rangle$ and therefore increasing $\langle \alpha \rangle$.

Figures 11–13 show the results of Monte Carlo experiments for the system driven by colored noise. In each curve the time-evolving behavior of the second-order moment is depicted for various values of the correlation time τ_c. All three figures show essentially the same behavior for the

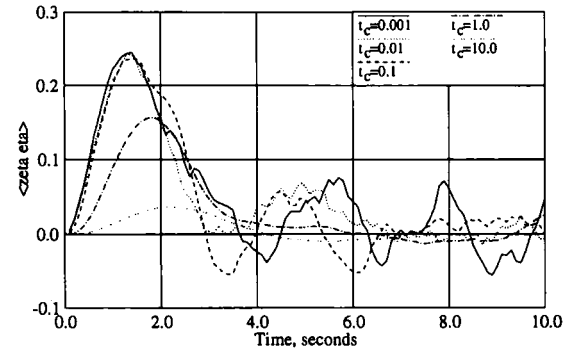

Fig. 12. $\langle \zeta \eta \rangle$ versus time for different values of τ_c, of $\gamma = 1.0$, $\alpha = 0.1$, calculated by Monte Carlo simulation.

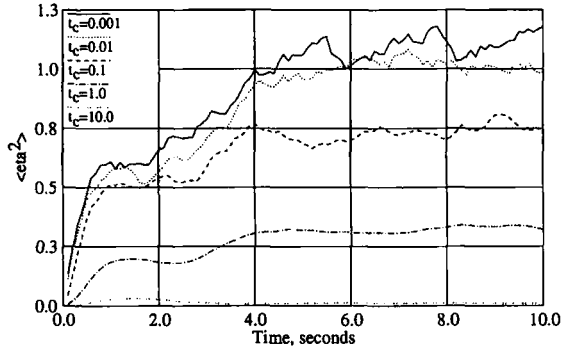

Fig. 13. $\langle \eta^2 \rangle$ versus time for different values of τ_c, $\gamma = 1.0$, $\alpha = 0.1$, calculated by Monte Carlo simulation.

case of $\tau_c = 0.001$ and $\tau_c = 0.01$ as for the white noise case, Figs. 1–3. For the case of $\tau_c = 0.1$, the effect of the correlated nature of the noise is noticeable, but perhaps acceptable for some applications. It is clear that when the correlation time becomes greater than 0.1, the results differ significantly from the white noise case. This is as one would expect: at $\tau_c > 0.1$ the correlation time becomes comparable to the natural period of the oscillator, which is about 1 sec. The correlated nature of the noise appears as a effect of time scale τ_c on the time history of the correlated noise. If the time scale of the correlation is much smaller than the natural period of the oscillator, then the oscillator cannot respond to this effect. However, as τ_c approaches the natural period of the oscillator, the oscillator can be, and is, affected.

4. CONCLUSIONS

It can be seen from the figures that the agreement between the results of the various methods of solution is good. The trends observed are not surprising. The further the parameters of the system are from violating the assumptions of the analysis, the better the analytical results agreed with those of the Monte Carlo experiment. It is also seen that, by and large, the higher-order analysis is more accurate. One interesting point is that the analytical techniques are consistently, albeit slightly, greater than the Monte Carlo experiments. This conservative nature of the analytical method should be noted in applying it.

The method can be applied to systems where the three basic conditions of the expansion are met: that the intensity of the forcing function α be of order smaller than unity, the damping of the system be of order at

least unity, and the forcing function be essentially uncorrelated in time, i.e., correlation time $\tau_c < 0.1$.

This technique is a valid analytical tool. It is well suited to studying the behavior of a system under a variety of conditions as long as the initial assumption that $\alpha \ll 1$ is not violated. The results it gives are accurate and computationally fast enough to embed such a model as an element of a large model.

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